

# DERIVATIONS OF NEGATIVE DEGREE ON QUASIHOMOGENEOUS ISOLATED COMPLETE INTERSECTION SINGULARITIES

MICHEL GRANGER AND MATHIAS SCHULZE

**ABSTRACT.** J. Wahl conjectured that every quasihomogeneous isolated normal singularity admits a positive grading for which there are no derivations of negative weighted degree. We confirm his conjecture for quasihomogeneous isolated complete intersection singularities of either order at least 3 or embedding dimension at most 5. For each embedding dimension larger than 5 (and each dimension larger than 3), we give a counter-example to Wahl's conjecture.

## CONTENTS

1. Introduction	1
2. Graded analytic algebras	3
3. Negative derivations	5
4. ICIS of embedding dimension 5	9
5. Counter-examples	10
References	11

## 1. INTRODUCTION

By a singularity we mean a quotient  $A$  of a convergent power series ring over a valued field  $K$  of characteristic zero (see §2). We use the acronym *negative derivation* for a derivation of negative weighted degree on a quasihomogeneous singularity. The question of existence of such negative derivations has important consequences in rational homotopy theory (see [Mei82, Thm. A]) and in deformation theory (see [Wah82, Thm. 3.8]).

By a result of Kantor [Kan79], quasihomogeneous curve and hypersurface singularities do not admit any negative derivations. J. Wahl [Wah82, Thm. 2.4, Prop. 2.8] reached the same conclusion in (the much deeper) case of quasihomogeneous normal surface singularities. Motivated by his cohomological characterization of projective space in [Wah83a], he formulates the following conjecture in [Wah83b, Conj. 1.4].

**Conjecture (Wahl).** *Let  $R$  be a normal graded ring, with isolated singularity. Then there is a normal graded  $\bar{R}$ , with  $\hat{R} \cong \hat{\bar{R}}$ , so that  $\bar{R}$  has no derivations of negative weight.*

In case  $R$  is a graded normal locally complete intersection with isolated singularity,  $\hat{R}$  becomes a quasihomogeneous normal isolated complete intersection singularity (ICIS) and Wahl's conjecture can be rephrased as follows (see Lemma 2.1 and Remark 2.3).

---

*Date:* June 27, 2014.

*1991 Mathematics Subject Classification.* 13N15, 14M10 (Primary) 14H20 (Secondary).

*Key words and phrases.* complete intersection, derivation, singularity.

The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n° PCIG12-GA-2012-334355.

**Conjecture** (Wahl, ICIS case). *Any quasihomogeneous normal ICIS has no negative derivations with respect to some positive grading.*

For quasihomogeneous normal ICIS, there is an explicit description of all derivations due to Kersken [Ker84]. Based on this description, we prove our main

**Theorem 1.1.** *For any quasihomogeneous normal ICIS of order at least 3 there are no negative derivations with respect to any positive grading.*

*Proof.* This follows from Corollary 3.4 and Proposition 3.8.  $\square$

Our investigations lead to a family of counter-examples to Wahl's Conjecture. In order to describe it, we first fix our notation. A quasihomogeneous singularity can be represented as

$$(1.1) \quad A = P/\mathfrak{a}, \quad \mathfrak{a} = \langle g_1, \dots, g_t \rangle \subseteq K\langle\langle x_1, \dots, x_n \rangle\rangle =: P$$

where  $g_1, \dots, g_t$  are homogeneous polynomials of degree  $p_i := \deg(g_i)$  with respect to weights  $w_1, \dots, w_n \in \mathbb{Z}_+$  on the variables  $x_1, \dots, x_n$  (see §2). We order these weights and degrees decreasingly as

$$(1.2) \quad \begin{aligned} w_1 &\geq \dots \geq w_n > 0, \\ p_1 &\geq \dots \geq p_t. \end{aligned}$$

*Example 1.2.* Let  $n \geq 6$  and pick  $c_7, \dots, c_n \in K \setminus \{1\}$  pairwise different such that  $c_i^9 + 1 \neq 0$  for all  $i$ . Assigning weights  $8, 8, 5, 2, \dots, 2$  to the variables  $x_1, \dots, x_n$ , the equations

$$(1.3) \quad \begin{aligned} g_1 &:= x_1 x_4 + x_2 x_5 + x_3^2 - x_4^5 + \sum_{i=7}^n x_i^5 \\ g_2 &:= x_1 x_5 + x_2 x_6 + x_3^2 + x_6^5 + \sum_{i=7}^n c_i x_i^5 \end{aligned}$$

define a quasihomogeneous complete intersection  $A$  as in (1.1) with isolated singularity. On  $A$  there is a derivation

$$(1.4) \quad \eta := \begin{vmatrix} \partial_1 & \partial_2 & \partial_3 \\ x_4 & x_5 & 2x_3 \\ x_5 & x_6 & 2x_3 \end{vmatrix} = 2x_3(x_5 - x_6)\partial_1 - 2x_3(x_4 - x_5)\partial_2 + (x_4 x_6 - x_5^2)\partial_3$$

of degree  $-1$ . We work out the details of this example in §5.

We show that Example 1.2 gives a counter-example to the ICIS case of Wahl's conjecture of minimal embedding dimension  $n = 6$ .

**Theorem 1.3.** *Exactly up to embedding dimension 5, all quasihomogeneous ICIS have no negative derivations with respect to some positive grading.*

*Proof.* This follows from Kantor [Kan79], [Wah82, Thm. 2.4, Prop. 2.8], Proposition 4.2, Example 1.2 and Corollary 3.4.  $\square$

As a consequence of our arguments we obtain a simple special case of the following conjecture due to S. Halperin.

**Conjecture** (Halperin). *On any graded zero-dimensional complete intersection there are no negative derivations.*

The following result bounds the degree of negative derivations (see also [Ale91, Prop.]). The bound does not require a complete intersection hypothesis and it is independent of further hypotheses as for instance in [Hau02, Thm. 2].

**Proposition 1.4.** *For any quasihomogeneous zero-dimensional singularity  $A$  as in (1.1) there are no derivations of degree strictly less than  $p_n - p_1$ . In particular, Halperin's conjecture holds true if  $p_1 = p_n$ .*

*Proof.* As  $A$  is assumed to be zero-dimensional, condition  $\mathfrak{A}(k)$  on page 4 must hold true for all  $k = 1, \dots, n$ . Then the claim follows from Remark 3.6 and Lemma 3.7.  $\square$

*Acknowledgments.* The second author would like to thank the LAREMA at the University of Angers for providing financial support and a pleasant working atmosphere during his research visit in February 2014.

## 2. GRADED ANALYTIC ALGEBRAS

Consider a (local) analytic algebra  $A = (A, \mathfrak{m}_A)$  over a (possibly trivially) valued field  $K$  of characteristic zero. We assume in addition that  $A$  is non-regular and can be represented as a quotient  $A = P/\mathfrak{a}$  of a convergent power series ring  $P := K\langle\langle x_1, \dots, x_n \rangle\rangle \supseteq \mathfrak{a}$ . In the sequel such an  $A$  will be referred to as a *singularity*. We choose  $n$  minimal such that  $n = \text{embdim } A$  and set  $d := \dim A$ .

A  $K_+$ -grading on  $A$  is given by a *diagonalizable derivation*  $\chi \in \text{Der}_K A =: \Theta_A$  which means that  $\mathfrak{m}_A$  is generated by eigenvectors  $x_1, \dots, x_n$  (see [SW73, (2.2),(2.3)]). Such a derivation is also called an *Euler derivation*. We refer to  $w_1, \dots, w_n$  defined by  $w_i := \chi(x_i)/x_i$  as the *eigenvalues* of  $\chi$ . More generally, we call  $\chi$ -eigenvectors  $f \in A$  ( $\chi$ -)homogeneous and define their ( $\chi$ -)degree to be the corresponding eigenvalue denoted by  $\deg(f) := \chi(f)/f \in K$ . We denote by  $A_a$  the  $K$ -vector space of all such eigenvector  $f \in A$  with  $\deg(f) = a$ . This defines a  $K$ -subalgebra

$$(2.1) \quad \bar{A} := \bigoplus_{a \in K} A_a \subset A \subset \hat{A}.$$

The derivation  $\chi \in \Theta_A$  lifts to  $\chi \in \Theta_P := \text{Der}_K P$  (see [SW73, (2.1)]). In particular,  $P$  is  $K_+$ -graded and  $\mathfrak{a} \subseteq P$  is a  $\chi$ -invariant ideal and hence ( $\chi$ -)homogeneous (see [SW73, (2.4)]). Pick homogeneous  $g_1, \dots, g_t \in \mathfrak{a}$  inducing a  $K$ -vector space basis of  $\mathfrak{a}/\mathfrak{m}_A \mathfrak{a}$ . Then  $\mathfrak{a} = \langle g_1, \dots, g_t \rangle$  by Nakayama's Lemma. We set  $p_i := \deg(g_i)$  ordered as in (1.2). To summarize, we can write  $A$  as in (1.1).

A  $K_+$ -grading is called a *positive grading* if  $w_i \in \mathbb{Z}_+$  for all  $i = 1, \dots, n$  (see [SW73, §3, Def.]). We call  $A$  *quasihomogeneous* if it admits a positive grading. In this case, we shall always normalize  $\chi$  to make the  $w_i$  coprime and order the variables according to (1.2). Positivity of weights enforces  $g_i \in \bar{P} = K[x_1, \dots, x_n]$  and that

$$(2.2) \quad \bar{A} = \bigoplus_{i \geq 0} A_i = \bar{P}/\bar{\mathfrak{a}}, \quad \bar{\mathfrak{a}} = \langle g_1, \dots, g_t \rangle \subseteq K[x_1, \dots, x_n] = \bar{P},$$

is a (positively) graded-local  $k$ -algebra with completion

$$(2.3) \quad \hat{\bar{A}} = \hat{A}$$

and graded maximal ideal  $\mathfrak{m}_{\bar{A}} = \bar{\mathfrak{m}}_A := \bigoplus_{i > 0} A_i$ . The preceding discussion enables us to reformulate Wahl's Conjecture in the language of Scheja and Wiebe as follows.

**Lemma 2.1.** *The following supplementary structures on a singularity  $A$  are equivalent:*

- (1) *an Euler derivation  $\chi$  on  $A$  with positive eigenvalues,*
- (2) *a positive grading on  $A$ ,*
- (3) *a positive grading on  $\hat{A}$ ,*
- (4) *a (positively) graded  $K$ -algebra  $\bar{A}$  such that  $\hat{\bar{A}} = \hat{A}$ .*

*Proof.* The equivalences of (1), (2), and (3) are due to Scheja and Wiebe (see [SW73, (2.2),(2.3)] and [SW77, (1.6)]). For the equivalence with (4), note that the obvious Euler derivation on a graded  $K$ -algebra  $\bar{A}$  lifts to an Euler derivation on the completion  $\hat{\bar{A}} = \hat{A}$ . The converse follows from (2.1), (2.2) and (2.3).  $\square$

Let us assume now that  $A$  is an isolated complete intersection singularity (ICIS). We may then take  $g_1, \dots, g_t$  to be a regular sequence and  $d + t = n$ . The isolated singularity hypothesis can be expressed in terms of the Jacobian ideal

$$(2.4) \quad J_A := \left\langle \left| \frac{\partial g}{\partial x_\nu} \right| \mid |\nu| = t \right\rangle \trianglelefteq A$$

of  $A$  as follows.

**Proposition 2.2.** *A complete intersection singularity  $A$  is isolated if and only if  $J_A$  is  $\mathfrak{m}_A$ -primary. An analogous statement holds for  $\bar{A}$ .*

*Proof.* We denote by  $\Omega_{A/k}^1$  the universally finite module of differentials of  $A$  over  $k$ . By the standard sequence

$$\mathfrak{a}/\mathfrak{a}^2 \longrightarrow A \otimes_P \Omega_{P/k}^1 \longrightarrow \Omega_{A/k}^1 \longrightarrow 0,$$

the Jacobian ideal  $J_A$  is the 0th Fitting ideal  $F_A^0 \Omega_{A/k}^1$ . By [SS72, (6.4),(6.9)], reducedness of  $A$  is equivalent to  $\text{rk } \Omega_{A/k}^1 = d$  and  $A_{\mathfrak{p}}$  is regular if and only if  $\Omega_{A_{\mathfrak{p}}/k}^1$  is free. Hence,  $A_{\mathfrak{p}}$  being regular is equivalent to  $\mathfrak{p} \not\supset F_A^0 \Omega_{A/k}^1 = J_A$  by [BH93, Lem. 1.4.9]. In particular,  $A$  having an isolated singularity means exactly that  $A/J_A$  is supported at  $\mathfrak{m}_A$  and hence that  $J_A$  is  $\mathfrak{m}_A$ -primary as claimed. The analogous statement for  $\bar{A}$  is proved similarly.  $\square$

*Remark 2.3.* Let  $A$  be a quasihomogeneous singularity. By (2.2),

$$(2.5) \quad J_{\bar{A}} := \bar{J}_A = \left\langle \left| \frac{\partial g}{\partial x_\nu} \right| \mid |\nu| = t \right\rangle \trianglelefteq \bar{A}$$

is the Jacobian ideal of  $\bar{A}$  defined analogous to (2.4). By (2.3),  $A$  is a complete intersection if and only if  $\bar{A}$  is locally a complete intersection (see [BH93, Def. 2.3.1, Ex. 2.3.21.(c)]). By Proposition 2.2,  $A$  is an ICIS if and only if  $J_A$  is  $\mathfrak{m}_A$ -primary. This is equivalent to  $J_{\bar{A}}$  being  $\mathfrak{m}_{\bar{A}}$ -primary. The latter is then equivalent to  $\bar{A}$  being locally a complete intersection with isolated singularity by (2.5) and Proposition 2.2. Complete intersections are Cohen–Macaulay and hence  $(S_2)$  so normality is equivalent to  $(R_1)$  by Serre’s Criterion (see [BH93, §2.3, Thm. 2.2.22]). Since  $d = \dim A = \dim \bar{A}$  by (2.3) (see [BH93, Cor. 2.1.8]), normality for both  $A$  and  $\bar{A}$  reduces to  $d \geq 2$ .

Scheja and Wiebe [SW77, (3.1)] (see also [Sai71, Satz 1.3]) proved that any  $K_+$ -graded ICIS is quasihomogeneous unless  $t = 1$  and  $g_1 \notin \mathfrak{m}_P^3$ . Their starting point (see [SW77, (2.5)] and [Sai71, Lem. 1.5]) is that  $A$  being an ICIS implies, by Proposition 2.2, that for each  $k = 1, \dots, n$  one of the following two conditions must hold true.

$\mathfrak{A}(k)$  For some  $m \geq 2$  and  $1 \leq j \leq t$ , the monomial  $x_k^m$  occurs in  $g_j$ .

$\mathfrak{B}(k)$  For some pairwise different  $1 \leq \nu_1, \dots, \nu_t \leq n$ , each  $g_j$  contains a monomial  $x_k^{m_j} x_{\nu_j}$  for some  $m_j \geq 1$ .

The following result gives numerical constraints for  $A$  to be a quasihomogeneous ICIS.

**Lemma 2.4.** *If  $A$  is a quasihomogeneous ICIS then*

$$(2.6) \quad p_1 + \dots + p_j \geq w_1 + \dots + w_j + j$$

for all  $j = 1, \dots, t$ .

*Proof.* We proceed by induction on  $j$ . Assume that  $p_1 + \dots + p_{j-1} \geq w_1 + \dots + w_{j-1} + j - 1$  but  $p_1 + \dots + p_j \leq w_1 + \dots + w_j + j - 1$ . Then  $p_j \leq w_j$  and hence  $g_i = g_i(x_{j+1}, \dots, x_n)$  for all  $i = j, \dots, n$ . Then  $J_A$  maps to zero in

$$A/\langle x_{j+1}, \dots, x_n \rangle = K\langle x_1, \dots, x_j \rangle / \langle g_1, \dots, g_{j-1} \rangle$$

and hence  $J_A$  cannot be  $\mathfrak{m}_A$ -primary as required by Proposition 2.2.  $\square$

### 3. NEGATIVE DERIVATIONS

Let  $A$  be a quasihomogeneous singularity as in §2. The target of our investigations is the positively graded  $A$ -module  $\Theta_A = \text{Der}_K A$  of  $K$ -linear derivations on  $A$ . More precisely, we are concerned with the question whether its negative part

$$\Theta_{A, < 0} = \Theta_{\bar{A}, < 0} = \bigoplus_{i < 0} \Theta_{A, i}$$

is trivial. A priori this condition depends on the choice of a grading. In Proposition 3.1 below, we shall prove the independence of this choice for a general singularity under a strong hypothesis satisfied in the ICIS case (see Corollary 3.4). To this end, we write (see [SW73, (2.1)])

$$(3.1) \quad \Theta_A = \Theta_{\mathfrak{a} \subset P} / \mathfrak{a} \Theta_P$$

as a quotient of a  $(k, P)$ -Lie algebra

$$\Theta_{\mathfrak{a} \subset P} := \{ \delta \in \Theta_P \mid \delta \mathfrak{a} \subset \mathfrak{a} \} \supseteq \mathfrak{a} \Theta_P$$

of logarithmic derivations along  $\mathfrak{a}$  by the  $(k, P)$ -Lie ideal  $\mathfrak{a} \Theta_P$ .

**Proposition 3.1.** *Let  $A$  be a quasihomogeneous singularity with positive grading given by  $\chi$  and assume that*

$$(3.2) \quad \Theta_{\mathfrak{a} \subset P} = P\chi + \Theta'_P + \mathfrak{a} \Theta_P,$$

$$(3.3) \quad \text{for some } \Theta'_P \subset \mathfrak{m}_P^2 \Theta_P.$$

*Then the condition  $\Theta_{A, < 0} = 0$  and the  $p_1, \dots, p_t$  in (1.2) are independent of the chosen positive grading.*

*Proof.* Consider a second positive grading with corresponding Euler derivation  $\chi'$  (see Lemma 2.1). By (3.1) and (3.2), any  $\delta \in \Theta_A$  lifts to an element of  $\Theta_{\mathfrak{a} \subset P}$  of the form

$$(3.4) \quad \delta = c\chi + \delta_+, \quad \delta_+ = a\chi + \eta, \quad c \in K, \quad a \in \mathfrak{m}_P, \quad \eta \in \Theta'_P,$$

denoted by the same symbol. By (3.3) and the Leibniz rule,

$$(3.5) \quad \chi \mathfrak{m}_P^k \subset \mathfrak{m}_P^k, \quad \delta_+ \mathfrak{m}_P^k \subset \mathfrak{m}_P^{k+1}$$

for all  $k \geq 1$ . Specializing to  $\delta = \chi$ , this implies that  $\chi_+ = 0$  and  $\chi' = c\chi$  on  $\mathfrak{m}_A / \mathfrak{m}_A^2 = \mathfrak{m}_P / \mathfrak{m}_P^2$  and hence  $c = 1$  by the definition of a positive grading and our normalization of weights.

Using (3.1), we equip  $\Theta_A$  with the decreasing  $\mathfrak{m}_P$ -adic filtration  $F^\bullet$  induced from  $\Theta_P$  which is defined as follows

$$F^k \Theta_A = (\Theta_{\mathfrak{a} \subset P} \cap \mathfrak{m}_P^k \Theta_P) / (\mathfrak{a} \Theta_P \cap \mathfrak{m}_P^k \Theta_P).$$

Due to (3.3), (3.4) and (3.5) this is a filtration by  $(k, P)$ -Lie ideals and

$$\delta_+ F^k \Theta_A \subset F^{k+1} \Theta_A$$

for the adjoint action of  $\delta_+$ . Therefore, for any  $k \geq 1$ , the adjoint action of  $\chi' = \chi + \chi_+$  on the truncation

$$F^{\leq k} \Theta_A := \Theta_A / F^{k+1} \Theta_A$$

is triangularizable with semisimple part equal to that of  $\chi$ . Thus,  $\chi'$  and  $\chi$  have the same eigenvalues on  $F^{\leq k} \Theta_A$  for any  $k \geq 1$ . The first claim then follows by choosing  $k$  sufficiently large. A similar argument yields the second claim.  $\square$

For a Gorenstein singularity  $A$ , there is a natural way to produce elements of  $\Theta_A$ . The  $A$ -submodule  $\Theta'_A \subset \Theta_A$  of *trivial derivations* is by definition the image of the inclusion

$$(3.6) \quad \Omega_{A/K}^{d-1} \hookrightarrow \omega_{A/K}^{d-1} = \text{Hom}_A(\Omega_{A/K}^1, \omega_{A/K}^d) = \Theta_A \otimes_A \omega_{A/K}^d \cong \Theta_A.$$

We return to the case of an ICIS singularity  $A$ . For  $1 \leq \nu_0 < \dots < \nu_t \leq n$  with complementary indices  $1 \leq \mu_1 < \dots < \mu_{d-1} \leq n$ , the lift to  $P$  of the image of  $dx_{\mu_1} \wedge \dots \wedge dx_{\mu_{d-1}}$  can be written (up to sign) explicitly as

$$(3.7) \quad \delta_\nu := \begin{vmatrix} \partial_{\nu_0} & \dots & \partial_{\nu_t} \\ \partial_{\nu_0} g_1 & \dots & \partial_{\nu_t} g_1 \\ \vdots & & \vdots \\ \partial_{\nu_0} g_t & \dots & \partial_{\nu_t} g_t \end{vmatrix}.$$

Note that

$$(3.8) \quad \deg \delta_\nu = p_1 + \dots + p_t - w_{\nu_0} - \dots - w_{\nu_t},$$

$$(3.9) \quad \delta_\nu g_j = 0$$

for all  $j = 1, \dots, t$  and  $\nu$ . Consider the  $P$ -module

$$(3.10) \quad \Theta'_P := \langle \delta_\nu \mid 1 \leq \nu_0 < \dots < \nu_t \leq n \rangle_P \subset \Theta_P.$$

The key to our investigations is the following result due to Kersken [Ker84, (5.2)]. From now on we assume in addition that  $A$  is quasihomogeneous and normal, that is,  $\dim A \geq 2$ .

**Theorem 3.2** (Kersken). *Let  $A$  be a quasihomogeneous normal ICIS. Then the module  $\Theta_A$  of  $K$ -linear derivations on  $A$  is generated by the Euler derivation  $\chi$  and the trivial derivations  $\Theta'_A$ .*

Although Kersken only states that  $\Theta'_A$  is minimally generated by the  $\delta_\nu$  in (3.7), his arguments show that together with  $\chi$  they form a minimal set of generators of  $\Theta_A$ . We denote by  $\mu(-)$  the minimal number of generators.

**Corollary 3.3.** *Let  $A$  be quasihomogeneous normal ICIS. Then  $\Theta_A$  is minimally generated by the Euler derivation  $\chi$  and the trivial derivations  $\delta_\nu$  in (3.7). In particular,*

$$\mu(\Theta_A) = \binom{n}{t+1} + 1.$$

*Proof.* Since the case  $d = 2$  is covered by [Wah87, Prop. 1.12], we may assume that  $d \geq 3$ . In this case, the inclusion (3.6) fits into the following commutative diagram with exact

rows and columns (see [Ker84, Proof of (4.8)] or [Wah87, Prop. 1.7]).

$$(3.11) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^1_{\mathfrak{m}_A}(\Omega_{A/K}^d) & \xrightarrow[\cong]{\chi} & H^1_{\mathfrak{m}_A}(\Omega_{A/K}^{d-1}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \omega_{A/K}^d & \xrightarrow{\chi} & \omega_{A/K}^{d-1} & \xrightarrow{\chi} & \omega_{A/K}^{d-2} \\ & & \uparrow & & \uparrow & & \uparrow \cong \\ 0 & \longrightarrow & \Omega_{A/K}^d & \xrightarrow{\chi} & \Omega_{A/K}^{d-1} & \xrightarrow{\chi} & \Omega_{A/K}^{d-2} \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

It follows that

$$\chi(\omega_{A/K}^{d-1}) \cong \chi(\Omega_{A/K}^{d-1}) \cong \Omega_{A/K}^{d-1} / \chi(\Omega_{A/K}^d)$$

where  $\chi(\Omega_{A/K}^d) \subset \mathfrak{m}_A \Omega_{A/K}^{d-1}$  and hence

$$\mu(\chi(\Omega_{A/K}^{d-1})) = \mu(\Omega_{A/K}^{d-1}) = \mu(\Theta'_A).$$

Now the middle row of (3.11) yields an exact sequence

$$0 \longrightarrow A \xrightarrow{\chi} \Theta_A \longrightarrow \chi(\omega_{A/K}^{d-1}) \otimes (\omega_{A/K}^d)^{-1} \longrightarrow 0$$

Since  $\chi \notin \mathfrak{m}_A \Theta_A$ , the claim follows.  $\square$

Note that  $\Theta'_P$  in (3.10) satisfies (3.3) due to (3.7) unless  $t = 1$  and  $g_1 \notin \mathfrak{m}_P^3$ . As a consequence of Proposition 3.1 and Theorem 3.2 we therefore obtain the following result. It is crucial for Example 1.2 to be a counter-example to Wahl's Conjecture.

**Corollary 3.4.** *Let  $A$  be a quasihomogeneous normal ICIS. Unless  $t = 1$  and  $g_1 \notin \mathfrak{m}_P^3$ , the condition  $\Theta_{A,<0} = 0$  and the  $p_1, \dots, p_t$  in (1.2) are independent of the choice of a positive grading.*  $\square$

We shall now derive numerical constraints for minimal negative trivial derivations. To this end, suppose that  $0 \neq \eta \in \Theta_{A,<0}$ . For degree reasons (see (1.2)),  $\eta$  can be written as

$$(3.12) \quad \eta = q_1 \partial_1 + \dots + q_n \partial_n, \quad q_i = q_i(x_{i+1}, \dots, x_n)$$

By Theorem 3.2, we may assume that  $\eta = \delta_\nu \neq 0$  is a trivial derivation as in (3.7). By (1.2) and (3.8), we may further assume that  $\nu_i = i + 1$  for all  $i = 0, \dots, t$  and hence  $q_i = 0$  for all  $i > t + 1$ . The preceding arguments combined with (3.8) and (3.9), can be summarized as follows.

**Lemma 3.5.** *Let  $A$  be a quasihomogeneous normal ICIS. Then, for all  $\eta \in \Theta_{A,<0}$  and all  $j = 1, \dots, t$ , we have*

$$(3.13) \quad \eta g_j = 0.$$

*If  $\Theta_{A,<0} \neq 0$  then there is a derivation  $0 \neq \eta \in \Theta_{A,<0}$  as in (3.12) with  $q_i = 0$  for all  $i > t$ . It gives rise to an inequality*

$$(3.14) \quad p_1 + \dots + p_t < w_1 + \dots + w_{t+1}.$$



*Remark 3.6.* For degree reasons (see (1.2)), the identity (3.13) holds true for any  $\eta \in \Theta_{A, < p_t - p_1}$  and any quasihomogeneous singularity  $A$  as in (1.1).

We now link the conditions  $\mathfrak{A}(k)$  and  $\mathfrak{B}(k)$  from page 4 to the existence of a negative derivation.

**Lemma 3.7.** *Assume that  $\eta \in \Theta_{A, < 0}$  as in (3.12) with  $q_i = 0$  for all  $i \in I \supset \{0, \dots, k-1\}$  satisfies (3.13) for all  $j = 1, \dots, t$  and that  $\mathfrak{A}(k)$  holds true. Then there is a  $\chi$ -homogeneous coordinate change preserving the preceding conditions which makes  $q_k = 0$ .*

*Proof.* As  $\mathfrak{A}(k)$  holds by hypothesis, there is a  $g := g_j$  containing a monomial  $x_k^m$ ,  $m > 1$ . Expanding respect to  $x_k$ ,

$$g = \sum_{j=0}^m t_j x_k^{m-j}, \quad t_j = t_j(x_1, \dots, \widehat{x_k}, \dots, x_n).$$

We may normalize  $g$  such that  $t_0 = \frac{1}{m}$ . Note that  $t_j$  is homogeneous of degree  $j \cdot w_k$  and, in particular,  $\deg(t_1) = \deg(x_k)$ . Then, ordering terms according to  $i = k$  or  $i > k$ , (3.13) becomes

$$\begin{aligned} (3.15) \quad 0 = \eta(g) &= \sum_{i \geq k} \sum_{j=0}^m q_i \partial_i (t_j x_k^{m-j}) \\ &= \sum_{j=1}^m \left( (m-j+1) q_k t_{j-1} + \sum_{i > k} q_i \partial_i (t_j) \right) x_k^{m-j}. \end{aligned}$$

By (3.12), all  $q_i$ ,  $i \geq k$ , are independent of  $x_k$ . Thus, using  $t_0 = \frac{1}{m}$ , the coefficient equation of  $x_k^{m-1}$  in (3.15) reads

$$q_k + \sum_{i > k} q_i \partial_i (t_1) = 0$$

and  $\eta$  can be rewritten as

$$(3.16) \quad \eta = \sum_{i > k} q_i \cdot (\partial_i - \partial_i(t_1) \partial_k).$$

The  $\chi$ -homogeneous coordinate change

$$x'_i = \begin{cases} x_k + t_1, & \text{if } i = k, \\ x_i, & \text{otherwise,} \end{cases}$$

replaces  $\partial_i - \partial_i(t_1) \partial_k$  in (3.16) by  $\partial'_i$ , and thus  $q_k$  in (3.12) by 0.  $\square$

Our main technical result is the following

**Proposition 3.8.** *Let  $A$  be a quasihomogeneous normal ICIS such that  $\Theta_{A, < 0} \neq 0$ . Then  $\mathfrak{B}(k)$  holds true for some  $k \leq t$  after some  $\chi$ -homogeneous coordinate change. Each such  $k$  satisfies  $k \geq t - d + 2$  and  $g_k, \dots, g_t \notin \mathfrak{m}_P^3$ .*

*Proof.* Let  $0 \neq \eta \in \Theta_{A, < 0}$  be given by Lemma 3.5. For increasing  $k \geq 1$  with  $q_k \neq 0$ , we repeatedly apply Lemma 3.7 with  $I = \{1, \dots, k-1, t+2, \dots, n\}$  as long as  $\mathfrak{A}(k)$  holds. The procedure stops with

$$(3.17) \quad 0 \neq \eta = q_k \partial_k + \dots + q_{t+1} \partial_{t+1}, \quad q_k \neq 0,$$

for some  $k \leq t+1$  by choice of  $\eta$  and Lemma 3.7. Since  $\mathfrak{A}(k)$  fails,  $\mathfrak{B}(k)$  must hold true. In case  $k = t+1$  in (3.17), (3.13) becomes  $\partial_k g_j = 0$  for all  $j = 1, \dots, t$ . This would mean



that all  $g_j$  are independent of  $x_k$  in contradiction to the isolated singularity hypothesis. Therefore,  $k \leq t$  as claimed.

Combining (2.6) and (3.14), we obtain

$$(3.18) \quad p_j + \cdots + p_t + j \leq w_j + \cdots + w_{t+1}$$

for all  $j = 1, \dots, t$ . Using (1.2),  $\mathfrak{B}(k)$  and (3.18) for  $j = k$ , we compute

$$\begin{aligned} m_k w_k + \cdots + m_t w_t &\leq (m_k + \cdots + m_t) w_k \\ &= \deg(\partial_{\nu_k} g_k \cdots \partial_{\nu_t} g_t) \\ &= p_k + \cdots + p_t - w_{\nu_k} - \cdots - w_{\nu_t} \\ &\leq w_k + \cdots + w_{t+1} - k - w_{\nu_k} - \cdots - w_{\nu_t}. \end{aligned}$$

and hence

$$(m_k - 1)w_k + \cdots + (m_t - 1)w_t \leq w_{t+1} - k - w_{\nu_k} - \cdots - w_{\nu_t}.$$

By (1.2), this forces

$$(3.19) \quad \begin{aligned} m_k &= \cdots = m_t = 1, \\ w_{t+1} &\geq w_{\nu_k} + \cdots + w_{\nu_t} + k. \end{aligned}$$

In particular,

$$(3.20) \quad \nu_k, \dots, \nu_t \geq t + 2$$

and hence  $k \geq t - d + 2$ . □

#### 4. ICIS OF EMBEDDING DIMENSION 5

**Lemma 4.1.** *Let  $A$  be a quasihomogeneous normal ICIS such that  $\Theta_{A, < 0} \neq 0$ . Then  $\mathfrak{A}(k_1)$  and  $\mathfrak{B}(k_2)$  for  $\{k_1, k_2\} = \{1, 2\}$  is impossible.*

*Proof.* Assuming the contrary, one of the  $g_j$  has a monomial divisible by  $x_{k_1}^2$  by  $\mathfrak{A}(k_1)$  and each of the  $g_j$  has a monomial divisible by  $x_{k_2}$  by  $\mathfrak{B}(k_2)$ . In particular,

$$p_1 + \cdots + p_t \geq 2w_{k_1} + (t - 1)w_{k_2} \geq w_1 + \cdots + w_{t+1}$$

contradicting (3.14). □

**Proposition 4.2.** *For any quasihomogeneous ICIS  $A$  as in (1.1) with  $n = 5$  and  $t = 2$ , we have  $\Theta_{A, < 0} = 0$ .*

*Proof.* Assume that  $\Theta_{A, < 0} \neq 0$ . By Proposition 3.8 and Lemma 4.1, we must have  $\mathfrak{B}(1)$  and  $\mathfrak{B}(2)$ . Using (1.2), (3.19), and (3.20), we may write

$$\begin{aligned} g_1 &= x_1 x_4 + c_1 x_2^j x_{k_1} + \cdots \\ g_2 &= x_1 x_5 + c_2 x_2 x_{k_2} + \cdots \end{aligned}$$

with  $\{k_1, k_2\} = \{4, 5\}$  and  $c_1, c_2 \in K^*$ . As in the proof of Lemma 4.1, the inequality (3.14) can only hold true if  $j = 1$ . In this case,

$$A/(J_A + \langle x_3, \dots, x_n \rangle) = K \langle \langle x_1, x_2 \rangle \rangle / \left\langle \left| \frac{\partial g}{\partial(x_4, x_5)} \right| \right\rangle.$$

for degree reasons (see (1.2)), and hence  $J_A$  is not  $\mathfrak{m}_A$ -primary. This contradicts to the isolated singularity hypothesis. □

## 5. COUNTER-EXAMPLES

*Proof of Example 1.2.* The sequence  $g$  is clearly regular and defines a complete intersection as in (1.1). Note that  $\eta$  in (1.4) agrees with  $\eta = \delta_{1,2,3}$  in (3.12). Since  $\deg(g_1) = 10 = \deg(g_2)$ , (3.9) shows that  $\eta$  has negative degree  $\deg \eta = -1$ .

It remains to check that  $A$  has an isolated singularity, that is, the Jacobian ideal  $J_A$  from (2.4) is  $\mathfrak{m}_A$ -primary. To this end, we may assume that  $K = \bar{K}$  which enables us to argue geometrically on the variety

$$\bar{X} := \text{Spec } \bar{A} \subset \mathbb{A}_K^n$$

with  $\bar{A}$  as in (2.2) using the Nullstellensatz.

The ideal  $J_A$  is the image in  $A$  of the Jacobian ideal  $\bar{J}_g \subseteq \bar{P}$  of  $g$  generated by the  $2 \times 2$ -minors

$$M_{i,j} := \left| \frac{\partial g}{\partial(x_i x_j)} \right|$$

of the Jacobian matrix of  $g$  which reads

$$\frac{\partial g}{\partial x} = \begin{pmatrix} x_4 & x_5 & 2x_3 & x_1 - 5x_4^4 & x_2 & 0 & 5x_7^4 & \cdots & 5x_n^4 \\ x_5 & x_6 & 2x_3 & 0 & x_1 & x_2 + 5x_6^4 & 5c_7x_7^4 & \cdots & 5c_nx_n^4 \end{pmatrix}.$$

With this notation we have to show that

$$\text{Sing } \bar{X} = V(g, \bar{J}_g) = \{0\}.$$

Due to those  $2 \times 2$ -minors of  $\frac{\partial g}{\partial x}$  which involve only the columns  $3, 7, 8, 9, \dots, n$ , only one of components  $x_3, x_7, x_8, x_9, \dots, x_n$  of any  $x \in \text{Sing } \bar{X}$  can be non-zero. We may therefore reduce to the case  $n \leq 7$ .

Because of the 3rd column of  $\frac{\partial g}{\partial x}$ , we have  $\bar{J}_g \cap K[x_1, \dots, x_6] \supseteq x_3I$  where

$$I := \langle x_4 - x_5, x_5 - x_6, x_1 - x_2, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.$$

Note that  $V(I)$  is the  $x_3$ -axis which is not contained in  $V(g)$ . It follows that  $\text{Sing } \bar{X} \cap V(x_7)$  is contained in the hyperplane  $V(x_3)$ . Similarly because of the 7th column of  $\frac{\partial g}{\partial x}$  and setting  $c := c_7$ , we have  $\bar{J}_g \cap K[x_1, \dots, \hat{x}_3, \dots, x_7] \supseteq x_7I'$  where

$$I' := \langle cx_4 - x_5, cx_5 - x_6, cx_2 - x_1, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.$$

Using  $c^9 + 1 \neq 0$ , we find that  $V(I')$  is the  $x_7$ -axis and conclude  $\text{Sing } \bar{X} \cap V(x_3) \subset V(x_7)$  as before. Summarizing the two cases,  $\text{Sing } \bar{X}$  is in fact contained in  $V(x_3, x_7)$ .

Fix a point  $(x_1, x_2, 0, x_4, x_5, x_6, 0) \in \text{Sing } \bar{X}$ . Successively using the the equations

$$\begin{aligned} M_{1,2} &= x_4x_6 - x_5^2 = 0, \\ M_{2,5} &= x_1x_5 - x_2x_6 = 0, \\ g_2 &= x_1x_5 + x_2x_6 + x_6^5 = 0, \\ M_{4,5} &= x_1(x_1 - 5x_4^4) = 0, \\ M_{5,6} &= x_2(x_2 + 5x_6^4) = 0, \end{aligned}$$

we derive

$$x_4 = 0 \Rightarrow x_5 = 0 \Rightarrow x_2x_6 = 0 \Rightarrow x_6 = 0 \Rightarrow x_1 = x_2 = 0.$$

Similarly  $x_6 = 0$  leaves no possibility except  $x = 0$  and  $x_5 = 0$  reduces to one of these two cases by  $M_{1,2} = 0$ .

Assume now that  $x_4, x_5, x_6$  are all non zero. Then the minors  $M_{1,5}, M_{2,4}, M_{2,5}, M_{2,6}$  give equations

$$x_1x_4 = x_2x_5, \quad x_1 = 5x_4^4, \quad x_1x_5 = x_2x_6, \quad x_2 = -5x_6^4.$$

Substituting into  $g$ , we obtain

$$g_1 = 2x_1x_4 - x_4^5 = 9x_4^5, \quad g_2 = 2x_2x_6 + x_6^5 = -9x_6^5$$

and hence  $x_4 = x_6 = 0$  contradicting our assumption.  $\square$

## REFERENCES

- [Ale91] A. G. Aleksandrov, *Vector fields on a complete intersection*, Funktsional. Anal. i Prilozhen. **25** (1991), no. 4, 64–66. MR 1167721 (93d:14073) [1](#)
- [BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR MR1251956 (95h:13020) [2](#), [2.3](#)
- [Hau02] V. Hauschild, *Discriminants, resultants and a conjecture of S. Halperin*, Jahresber. Deutsch. Math.-Verein. **104** (2002), no. 1, 26–47. MR 1913265 (2003h:55018) [1](#)
- [Kan79] Jean-Michel Kantor, *Rectificatif à la note: “Dérivations sur les singularités quasi-homogènes: cas des courbes”*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 14, A697 (French). [1](#), [1](#)
- [Ker84] Masumi Kersken, *Reguläre Differentialformen*, Manuscripta Math. **46** (1984), no. 1–3, 1–25. MR 735512 (85j:14032) [1](#), [3](#), [3](#)
- [Mei82] W. Meier, *Rational universal fibrations and flag manifolds*, Math. Ann. **258** (1981/82), no. 3, 329–340. MR 649203 (83g:55009) [1](#)
- [Sai71] Kyoji Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen*, Invent. Math. **14** (1971), 123–142. MR 0294699 (45 #3767) [2](#)
- [SS72] Günter Scheja and Uwe Storch, *Differentielle Eigenschaften der Lokalisierungen analytischer Algebren*, Math. Ann. **197** (1972), 137–170. MR 0306172 (46 #5299) [2](#)
- [SW73] Günter Scheja and Hartmut Wiebe, *Über Derivationen von lokalen analytischen Algebren*, Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), Academic Press, London, 1973, pp. 161–192. MR 0338461 (49 #3225) [2](#), [2](#), [2](#), [3](#)
- [SW77] ———, *Über Derivationen in isolierten Singularitäten auf vollständigen Durchschnitten*, Math. Ann. **225** (1977), no. 2, 161–171. MR 0508048 (58 #22649) [2](#), [2](#)
- [Wah83a] J. M. Wahl, *A cohomological characterization of  $\mathbf{P}^n$* , Invent. Math. **72** (1983), no. 2, 315–322. MR 700774 (84h:14024) [1](#)
- [Wah83b] Jonathan M. Wahl, *Derivations, automorphisms and deformations of quasihomogeneous singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 613–624. MR 713285 (85g:14008) [1](#)
- [Wah87] ———, *The Jacobian algebra of a graded Gorenstein singularity*, Duke Math. J. **55** (1987), no. 4, 843–871. MR 916123 (89a:14042) [3](#)
- [Wah82] ———, *Derivations of negative weight and nonsmoothability of certain singularities*, Math. Ann. **258** (1981/82), no. 4, 383–398. MR 650944 (84h:14043) [1](#), [1](#)

M. GRANGER, UNIVERSITÉ D’ANGERS, DÉPARTEMENT DE MATHÉMATIQUES, LAREMA, CNRS UMR n°6093, 2 Bd LAVOISIER, 49045 ANGERS, FRANCE

*E-mail address:* [granger@univ-angers.fr](mailto:granger@univ-angers.fr)

M. SCHULZE, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KAISERSLAUTERN, 67663 KAISERSLAUTERN, GERMANY

*E-mail address:* [mschulze@mathematik.uni-kl.de](mailto:mschulze@mathematik.uni-kl.de)